Matrix Geometric Analysis of Congestion System using Phase-Type Distribution

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***Abstract*—: In this paper congestion of system is analyze through the continuous time queueing system. Congestion of a system can easily be avoided by applying the phase type distribution as an arrival and service processes. Erlang phase type distribution is used as an arrival and service processes and system is solved through the matrix geometric method. Analytical results are obtained for the mean flow time of the customers in the system for the different number of Erlang phases and compare the results with the hyperexponential distribution**

Keywords—: *Queuing System, Phase type distribution, Matrix Geometric Analysis, Mean number, Congestion*

I. Introduction

Queuing systems are models of systems providing service. Such a model may represent any system where jobs or customers arrive looking for service of some kind and depart after such service has been provided. We can model systems of this type as either single queues or a system of interconnected queues forming a queueing network. Some of the important applications of queuing modeled system are in communication systems, stocking systems production systems, information processing systems and transportation.

Congestion in systems leads to a loss of information and the system reaches a blocking state where it is not able to accept the information. Vector domain method is an efficient method to analyze system behavior by utilizing the features of matrices. Congestion in systems having phase-type distribution can be efficiently resolved through the vector domain method. There are many systems in which a congestion analysis is required.

They are used for modeling various random times, in particular, those which appear in manufacturing systems as processing times[1]. Balance between maximizing throughput & congestion control is achieved through the use of transmission control protocols & selective dropping of packets [2]. Analyze a sequence of stationary queueing networks by considering a sequence of congestion controls and Notion of utility optimization can be solved by considering queueing networks with end-to-end control [3]. Established congestion free and reliable data transfer in cable access networks and Use access network hybridized with mathematical queueing models to maintain the data packets in a scheduled manner [4]. Determined the optimal point at which congestion can be avoided in a manufacturing system after a series of iterations [5]. Priority-based and clustering-based approaches show enhanced performance as compared to other congestion avoidance approaches [6]. Formulate the delay-aware congestion control as a network utility maximization, which considers the link capacity and end-to-end delay as constraints [7].

The organization of this paper is as follows: Queueing System, Matrix Geometric method and Phase type distribution are briefly discussed in Sections 2, 3 and 4. In Section 5, phase type QBD models and its matrix geometric solutions are discussed. Finally, the results and conclusion are presents in Sections 6and 7 respectively.

II. TERMINOLOGIES

A. *Queuing System*

Stochastic modelling is one of the most important domains in the field of queueing theory. The concept of a queue which can be used to modelled many real systems. A queue is the main element which is used to hold the incoming and out coming customers before the served as shown in figure. 1. It is said that the users are served in the system by one or many servers [8]. Finally, a service discipline B needs to be specified. This can be first come first served (FCFS), last come first served (LCFS), processor sharing (PS), sometimes working with certain priorities or preemption rules. Normally we choose FCFS, meaning that the first user who arrives in the system will be the first to get access to a server.

Fig.1. Basic Queuing System

B. *Matrix Geometric Method (MGM)*

Matrix geometric method is a method for the analysis of quasi-birth–death processes, continuous-time Markov chain whose transition rate matrices with a repetitive block structure [11,12]. The method requires a transition rate matrix with tridiagonal block structure as follows

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$$
Q = \begin{pmatrix} B_{00} & B_{01} & & & & \\ B_{10} & A_1 & A_2 & & & \\ & A_0 & A_1 & A_2 & & \\ & & A_0 & A_1 & A_2 & \\ & & & A_0 & A_1 & A_2 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}
$$

Where each of B_{00}, B_{01}, B_{10}, A_0, A_1 and A_2 are matrices. This method should verify that the matrix satisfies requirements of QBD structure, verify that stability condition is satisfied, Use recursion to compute the R-matrix, Solve the set of equations to calculate p_0 and p_1, Use recursion $p_n = p_{n-1} R$ to find all other p_n vectors.

C. Phase Type Distribution

A phase-type distribution is a probability distribution constructed by a convolution or mixture of exponential distributions. It results from a system of one or more inter-related Poisson processes occurring in sequence, or phases. Consider a continuous-time Markov process with $m + 1$ states, where $m \geq 1$, such that the states $1,...,m$ are transient states and state 0 is an absorbing state. Further, let the process have an initial probability of starting in any of the $m + 1$ phases given by the probability vector (α_0,α) where α_0 is a scalar and α is a $1 \times m$ vector. The continuous phase-type distribution is the distribution of time from the above process's starting until absorption in the absorbing state [9,10]. This process can be written in the form of a transition rate matrix

$$
Q = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{S}^0 & S \end{bmatrix}
$$

III. SYSTEM MODEL

D. Phase-Type Distribution Arrival Process

A finite queue in which arrival process follows the phase-type distribution and service is exponentially distributed.Consider a finite queue in which arrivals follows the Erlang-r distribution with rate r¸ and the service process is exponentially distributed with rate μ as shown in figure.2.

The Markov chain and its resulting Finitesimal generator matrix are shown in Figures 3 and 4.

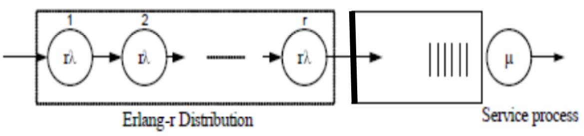


Fig.2. Queuing System model

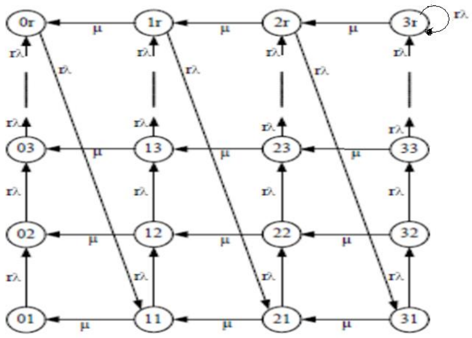


Fig.3. Markov chain

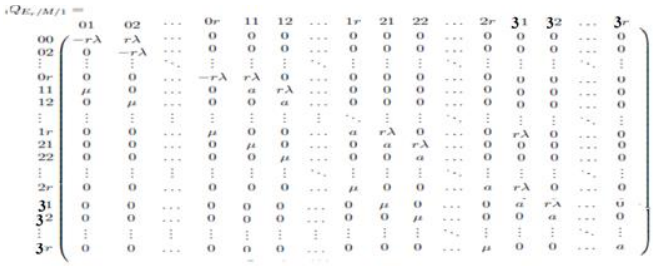


Fig.4. Finitesimal generator matrix

The system state space is represented by a (ij), i represents the number of customers in the system and j represents the current phase of the arrival. Partitioning of the levels in Markov chain and finitesimal generator matrix is given as in Figures 5 and 6.

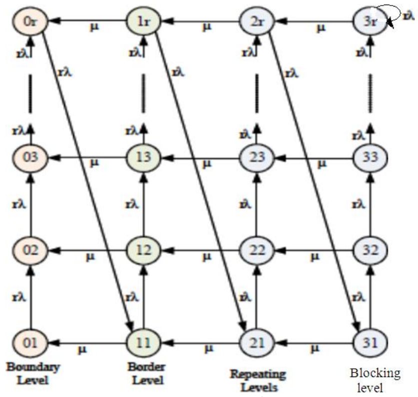


Fig.5. Partitioning of the levels

According to the level partitioning, different submatrices are developed as shown in Figure 7. The resulting form of the block form generator matrix is same as QBD process

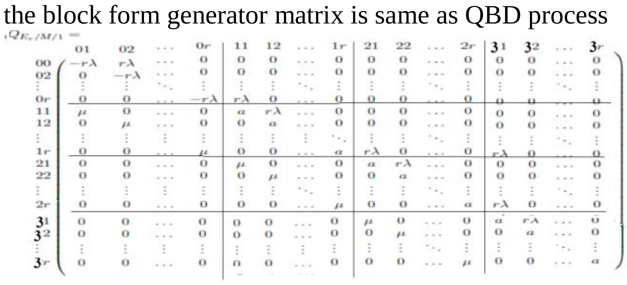


Fig.6. Partitioning of the levels

$$B_{00}=\begin{bmatrix}-r\lambda & r\lambda & \cdots & 0\\ 0 & -r\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -r\lambda\end{bmatrix}\quad B_{01}=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ r\lambda & 0 & \cdots & 0\end{bmatrix}\quad B_{10}=\begin{bmatrix}\mu & 0 & \cdots & 0\\ 0 & \mu & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ r\lambda & 0 & \cdots & \mu\end{bmatrix}$$

$$A_0=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ r\lambda & 0 & \cdots & 0\end{bmatrix}\quad A_1=\begin{bmatrix}-(r\lambda+\mu) & -r\lambda & \cdots & 0\\ 0 & -(r\lambda+\mu) & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(r\lambda+\mu)\end{bmatrix}\quad A_2=\begin{bmatrix}\mu & 0 & \cdots & 0\\ 0 & \mu & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ r\lambda & 0 & \cdots & \mu\end{bmatrix}$$

Fig.7. Submatrices

E. Phase-Type Distribution as A Service Process

A finite queue in which arrival process follows the Markovian distribution and service follows a phase-type distribution. Consider a finite queue with Poisson arrivals with rate λ, and the service process follows an Erlang-r distribution with rate rμ as shown in figure. 8

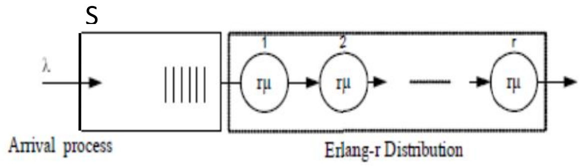


Fig.8. Queuing System model

The Markov chain and its resulting Finitesimal generator matrix are shown in Figures. 9 and 10.

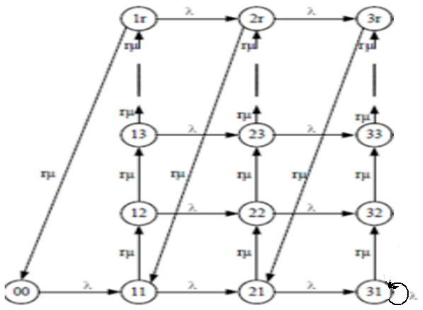


Fig.9. Markov chain

$$Q_{M/E_r/1}=\begin{array}{c|ccccccccccccc} & 00 & 11 & 12 & \ldots & 1r & 21 & 22 & \ldots & 2r & 31 & 32 & \ldots & 3r\\ \hline 00 & -\lambda & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\ 11 & 0 & a & r\mu & \ldots & 0 & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\ 12 & 0 & 0 & a & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1r & r\mu & 0 & 0 & \ldots & a & 0 & 0 & \ldots & \lambda & 0 & 0 & \ldots & 0\\ 21 & 0 & 0 & 0 & \ldots & 0 & a & r\mu & \ldots & 0 & \lambda & 0 & \ldots & 0\\ 22 & 0 & 0 & 0 & \ldots & 0 & 0 & a & \ldots & 0 & 0 & \lambda & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2r & 0 & r\mu & 0 & \ldots & 0 & 0 & 0 & \ldots & a & 0 & 0 & \ldots & \lambda\\ 31 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & a & r\mu & \ldots & 0\\ 32 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & a & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 3r & 0 & 0 & 0 & \ldots & 0 & r\mu & 0 & \ldots & 0 & 0 & 0 & \ldots & a\end{array}$$

Fig.10. Finitesimal generator matrix

Partitioning of the levels in Markov chain and finitesimal generator matrix are given as in Figures. 11 and 12.

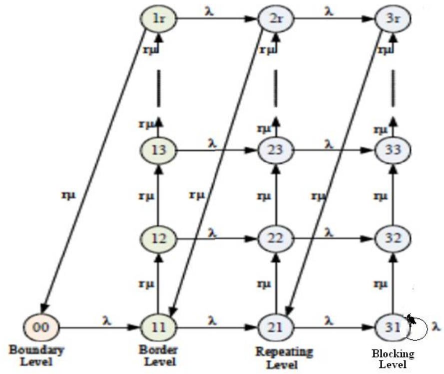


Fig.11. Partitioning of the levels

$$Q_{M/E_r/1}=\left(\begin{array}{c|cccc|cccc|cccc} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \hline 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ r\mu & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0\\ \hline 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0 & \lambda & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 & 0 & \lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a & 0 & 0 & \cdots & \lambda\\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a & r\mu & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & r\mu & 0 & \cdots & 0 & 0 & 0 & \cdots & a\end{array}\right)$$

Fig.12. Partitioning of the levels

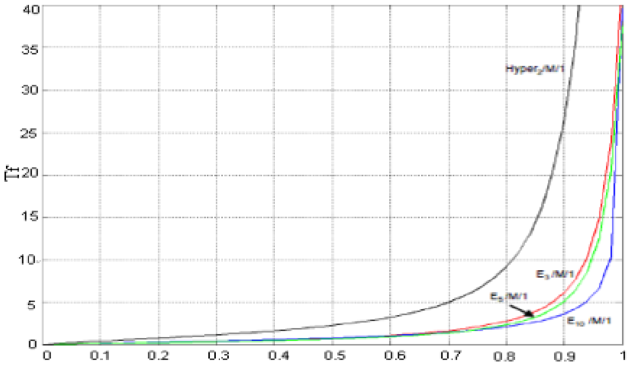
The resulting submatrices are shown in following Figure 13.

$$B_{00}=[-\lambda\ ]\quad B_{01}=[\lambda \quad 0 \quad \cdots \quad 0]\quad B_{10}=\begin{bmatrix}0\\ 0\\ \vdots\\ r\lambda\end{bmatrix}\quad A_0=\begin{bmatrix}\lambda & 0 & \cdots & 0\\ 0 & \lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$$

$$A_1=\begin{bmatrix}-(\lambda+r\mu) & -ru & \cdots & 0\\ 0 & -(\lambda+r\mu) & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(\lambda+r\mu)\end{bmatrix}\quad A_2=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ r\mu & 0 & \cdots & 0\end{bmatrix}$$

IV. RESULTS AND DISCUSSION

The mean flow time of the PH/M/1 queue plotted against the system utilization is given in Figure. 14 for Erlang and hyperexponeneial distribution. The mean flow time of the M/PH/1 queue plotted against the system utilization is given in Figure 15. Figure 15 shows the mean flow time of the customers in the system against the system utilization. It is observed that the mean flow time of customers in the system with hyperexponential distribution increases faster than the system with Erlang distribution. Here, it is also seen that the number of phases has significant effect on the system mean flow time. An increase in the Erlang phases, mean flow time increases slowly by varying the system utilization and finally increases to infinity when system utilization reaches near to the maximum utilization. Here again, it is seen the same behavior of the system for hyperexponential and Erlang distributions by varying the number of phases of the distribution.



. Fig. 14. Mean flow time in the system: PH/M/1 queue: E3, E5, E10, Hyper2, µ =1

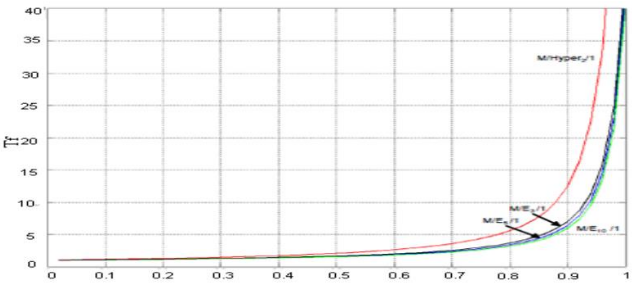


Fig. 15. Mean flow time in the system: M/PH/1 queue: E3, E5, E10 and H2

V. CONCLUSION

In this paper, we used a Phase-type distribution is used to avoid congestion in the system. The system is solved using Vector domain analysis is used to solve congestion problems in phase-type systems through flow time. The submatrices are used to trace the flow time of customers in the system. The mean flow time of the different systems having same phase type distributions as an arrival or service was analyzed.

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